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CROSS-CORRELATION COEFFICIENTS AND MODAL COMBINATION RULES FOR NON-CLASSICALLY DAMPED SYSTEMS

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SUMMARY

In stochastic analysis the knowledge of cross-correlation coefficients is required in order to combine the response of the modal Single-Degree-Of-Freedom (SDOF) oscillators for obtaining the nodal response. Moreover these coefficients play a fundamental role in the seismic analysis of structures when the response spectrum method is used. In fact they are used in some modal combination rules in order to obtain the maximum response quantities starting from the modal maxima. Herein a method for the evaluation of the cross-correlation coefficients for non-classically damped systems is presented. It is defined in the time domain instead of the frequency domain as usually encountered in the literature. Although non-classically damped structures possess complex eigenproperties, the great advantage in using this approach lies in the fact that the evaluation of these coefficients does not require complex quantities. Moreover a further particularization of the presented method allows a simple application of the spectrum analysis requiring only one response spectrum for an assigned damping ratio. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: cross-correlation coefficients; modal combination rule; non-classically damped systems; response spectrum method

1. INTRODUCTION

In many cases of practical interest the dynamic response of structures can be accurately determined once the undamped normal modes are evaluated. However this approach fails for particular structures, like the base-isolated buildings, the structures composed by several substructures and so on. Indeed in the latter cases the damping matrix has to be accounted for, which leads to complex eigenproperties.[1] In literature these structures are known as non-classically damped. The application of the traditional modal analysis to non-classically damped structures leads to a set of coupled second-order differential equations in the modal subspace. In order to uncouple these differential equations the complex eigenproperties have to be evaluated, obtaining

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a set of first-order uncoupled differential equations with complex coefficients.[2,3] It follows that the real response in the nodal subspace is evaluated by combining complex quantities.

The cross-correlation coefficients are extensively adopted in both stochastic and response spectrum analysis of structural systems under seismic input.[4] Indeed, by using these coefficients, the nodal response can be obtained by combining the response of SDOF oscillators, where the response has to be intended in terms of variances in the stochastic analysis and in terms of maximum value in the spectrum analysis. The cross-correlation coefficients are usually evaluated in the frequency domain for both classically damped and non-classically damped systems, involving, in this last case, complex quantities.[5–9]

Specifically, in the framework of the response spectrum method, the maximum response of a system is estimated by first determining the biggest values of the modal response, from the given response spectra of the given input, then by combining these maxima using an appropriate combination rule, whose coefficients are connected to the cross-correlation coefficients.[10] It has been recently recognized that for non-classically damped systems the proposed combination rules are quite different from the rules available for classically damped systems. Indeed, in obtaining the nodal response, the velocity response spectrum [11,12] or the cosine spectrum,[13] are necessary, in addition to the displacement response spectrum used in the combination rule of classically damped structures. Alternatively, the cross-correlation coefficients of non-classically damped systems are obtained by introducing corrective terms, accounting for the complex eigenproperties, in the coefficients available for classical damped ones.[14,15] These methods, beyond the difficulties connected to solving a complex eigenproblem, show another drawback when applied in the framework of the spectrum method. In fact they need the response spectra for those values of damping ratio which are obtained by the complex analysis. But the design maximum spectra, the so-called target ones, are usually given for one ($\xi = 0{\cdot}05$), or few ($\xi = 0{\cdot}01, 0{\cdot}02, 0{\cdot}05, 0{\cdot}2$) values of damping ratio and only some approximated relationships are given in order to find the spectra for different values of damping ratio.

In this paper a procedure for the evaluation of the cross-correlation coefficients in the time domain for non-classically damped systems is proposed. This procedure requires neither the evaluation of the complex eigenproperties of the structure nor the velocity or cosine spectra. Indeed, all the quantities necessary to evaluate the cross-correlation coefficients are real ones.

Moreover, the presented method has the advantage, with respect to those already presented in literature, that always the same results are obtained if the damping ratio is fixed equal to the value for which the response spectra are given. Hence, the above cited drawback, related to the approximated evaluation of the response spectrum different from the target one, is overcome. The last numerical example of the present work shows that the peak response spectra method can be easily applied to some kinds of structures, such as the isolated buildings, although it is not usually applied for such structures.

2. PRELIMINARY CONCEPTS

2.1. Equations of motion

Let us consider a discretized structural system with n dynamic degrees of freedom, subjected to the ground acceleration $\ddot{u}_g(t)$, whose dynamic equilibrium equation can be written as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = -\mathbf{M}\boldsymbol{\tau}\ddot{u}_g(t) \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the $(n \times n)$ mass and stiffness positive-definite matrices, respectively, $\mathbf{C}$ is the $n \times n$ damping positive or semipositive-definite matrix, $\mathbf{u}$, $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ are the $(n \times 1)$ vectors of nodal displacements, velocities and accelerations, respectively, and $\boldsymbol{\tau}$ is the $(n \times 1)$ vector of influence coefficients.

For both classically and non-classically damped structures it has been recognized that operating in the modal subspace is more convenient than in the nodal space.[1–3,16,17] In order to reduce the number of the variables, the following co-ordinate transformation is usually adopted:

$$\mathbf{u}(t) = \boldsymbol{\Phi}\mathbf{q}(t) \tag{2}$$

where $\mathbf{q}$ is the vector, or order $(m \times 1)$ (with $m \leqslant n$), or generalized coordinates and $\boldsymbol{\Phi}$ is the $(n \times m)$ modal matrix, normalized with respect to mass matrix and given by the solution of the following eigenproblem:

$$\mathbf{K}\boldsymbol{\Phi} = \mathbf{M}\boldsymbol{\Phi}\boldsymbol{\Omega}^2 \tag{3}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix listing the first m few natural circular frequencies.

By using equation (2), the differential equations of motion in the modal sub-space can be written as follows:

$$\ddot{\mathbf{q}}(t) + \boldsymbol{\Xi}\dot{\mathbf{q}}(t) + \boldsymbol{\Omega}^2\mathbf{q}(t) = \mathbf{p}\ddot{u}_{\mathrm{g}}(t) \tag{4}$$

where $\mathbf{p}$ is the vector of participation coefficients and $\boldsymbol{\Xi}$ is the generalized damping matrix, given respectively by

$$\mathbf{p} = -\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\tau, \quad \boldsymbol{\Xi} = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi} \tag{5}$$

where the apex T means transpose. For non-classically damped systems, the matrix $\boldsymbol{\Xi}$ is not diagonal. The relative maximum magnitude of the off-diagonal elements of $\boldsymbol{\Xi}$ with respect to the diagonal elements can be expressed by the following coupling index:[18]

$$\alpha = \max\left(\frac{\Xi_{ij}^2}{\Xi_{ii}\Xi_{ij}}\right), \quad i \neq j \tag{6}$$

which measures the degree of non proportionality of the damping. Notice that, for the semipositive definiteness of the damping matrix, the coupling index α is always less than one. By making use of the $2m$ state variable vector approach, equation (4) can be written as a set of $2m$ first-order differential equations as follows:

$$\dot{\mathbf{z}}(t) = \mathbf{D}\mathbf{z}(t) + \mathbf{V}\mathbf{p}\ddot{u}_{\mathrm{g}}(t) \tag{7}$$

where

$$\mathbf{z} = \begin{pmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ -\boldsymbol{\Omega}^2 & -\boldsymbol{\Xi} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{0}_m \\ \mathbf{I}_m \end{pmatrix} \tag{8}$$

In these relationships $\mathbf{I}_m$ is the identity matrix and $\mathbf{0}_m$ is the zero matrix both of order $(m \times m)$.

Solution of equation (7) can be pursued by solving the following eigenproblem:

$$\mathbf{D}\boldsymbol{\Psi} = \boldsymbol{\Psi}\boldsymbol{\Lambda} \tag{9}$$

where $\mathbf{\Lambda}$ is the diagonal matrix listing the complex eigenvalues Λ_k ($k = 1, 2, \ldots, m$) of the matrix $\mathbf{D}$ and $\mathbf{\Psi}$ is the complex eigenvector matrix orthonormalized with respect to the following matrix:

$$\mathbf{A} = \begin{pmatrix} \mathbf{\Xi} & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{0}_m \end{pmatrix} \tag{10}$$

The vector solution of equation (7) can be written as follows:[9,19]

$$\mathbf{z}(t) = \sum_{k=1}^{m} (\mathbf{L}_k \mathbf{p} d_k(t) + \mathbf{S}_k \mathbf{p} \dot{d}_k(t)) \tag{11}$$

where

$$\mathbf{L}_k = -2\mathrm{Re}(\Lambda_k^* \psi_k \psi_k^{\mathrm{T}})\mathbf{AV}; \quad \mathbf{S}_k = 2\mathrm{Re}(\psi_k \psi_k^{\mathrm{T}})\mathbf{AV} \tag{12}$$

where $\mathrm{Re}(\cdot)$ means real part of $(\cdot)$, the apex $*$ indicates the complex conjugate and ψ_k is the kth column of the matrix Ψ. In equation (11) d_k and $\dot{d}_k$ can be evaluated as the solution of the following differential equation:

$$\ddot{d}_k(t) + 2\beta_k \gamma_k \dot{d}_k(t) + \gamma_k^2 d_k(t) = \ddot{u}_{\mathrm{g}}(t); \quad d_k(0) = 0, \quad \dot{d}_k(0) = 0 \tag{13}$$

in which the natural damping ratio β_k and the natural frequency γ_k are obtained from the complex eigenvalue Λ_k as follows:

$$\beta_k = -\frac{\mathrm{Re}(\Lambda_k)}{\gamma_k}, \quad \gamma_k = |\Lambda_k| \tag{14}$$

$|(\cdot)|$ being the modulus of $(\cdot)$.

2.2. Covariance response for stationary stochastic input

Let us consider the excitation $\ddot{u}_{\mathrm{g}}(t)$ as a zero-mean Gaussian stationary process. Then, from a probabilistic point of view, the response is fully characterized by the covariance matrix, whose elements can be evaluated as follows:

$$E[\mathbf{z}^{[2]}(t)] = \sum_{j=1}^{m} \sum_{k=1}^{m} (\mathbf{L}_j \otimes \mathbf{L}_k)\mathbf{p}^{[2]} E[d_j(t) d_k(t)] + (\mathbf{L}_j \otimes \mathbf{S}_k)\mathbf{p}^{[2]} E[d_j(t)\dot{d}_k(t)] \\ + (\mathbf{S}_j \otimes \mathbf{L}_k)\mathbf{p}^{[2]} E[\dot{d}_j(t) d_k(t)] + (\mathbf{S}_j \otimes \mathbf{S}_k)\mathbf{p}^{[2]} E(\dot{d}_j(t)\dot{d}_k(t)] \tag{15}$$

in which $E[(\cdot)]$ means average of $(\cdot)$, $\otimes$ indicates the Kronecker product of two matrices (see Appendix) and the exponent into square brackets indicates the Kronecker power. i.e.

$$E[\mathbf{z}^{[2]}(t)] = E[\mathbf{z}(t) \otimes \mathbf{z}(t)] \tag{16}$$

It is easy to show that the quantity here expressed is the vectorized form of the covariance matrix.

Let us introduce the following cross-correlation coefficients:

$$\rho_{0,jk} = \frac{E[d_j(t) d_k(t)]}{\sigma_{d_j} \sigma_{d_k}}; \quad \rho_{1,jk} = \frac{E[d_j(t) \dot{d}_k(t)]}{\sigma_{d_j} \sigma_{\dot{d}_k}}; \quad \rho_{2,jk} = \frac{E[\dot{d}_j(t) \dot{d}_k(t)]}{\sigma_{\dot{d}_j} \sigma_{\dot{d}_k}} \tag{17}$$

in which $\sigma_{d_j} = \sqrt{E[d_j^2(t)]}$ and $\sigma_{\dot{d}_j} = \sqrt{E[\dot{d}_j^2(t)]}$ are the standard deviations of the displacement $d_j(t)$ and velocity $\dot{d}_j(t)$, respectively. By assuming that the relationship $\sigma_{\dot{d}_j} = \chi_j \sigma_{d_j}$ holds, where χ_j is a suitable coefficient, coincident with the natural frequency γ_j in the case of white noise input, then equation (15) can be rewritten as follows:

$$E[\mathbf{z}^{[2)}(t)] = \sum_{j=1}^{m} \sum_{k=1}^{m} \mathbf{R}_{jk} \mathbf{p}^{[2]} \sigma_{d_j} \sigma_{d_k} \tag{18}$$

where $\mathbf{R}_{jk}$ is a matrix of order $(2m)^2 \times (2m)^2$ given as follows:

$$\mathbf{R}_{jk} = (\mathbf{L}_j \otimes \mathbf{L}_k)\rho_{0,jk} + \chi_k(\mathbf{L}_j \otimes \mathbf{S}_k)\rho_{1,jk} + \chi_j(\mathbf{S}_j \otimes \mathbf{L}_k)\rho_{1,kj} + \chi_j\chi_k(\mathbf{S}_j \otimes \mathbf{S}_k)\rho_{2,jk} \tag{19}$$

Notice that the cross-correlation coefficients $\rho_{0,jk}$, $\rho_{1,jk}$ and $\rho_{2,jk}$ can be evaluated once the probabilistic parameters of the excitation are fixed. For example, for white noise excitations, the expressions of these coefficients have been provided in closed form by Der Kiureghian.[10]

2.3. Evaluation of the peak response by the response spectrum

A generic quantity $s(t)$ of interest, connected to the structural response, such as the stress at a point or the internal force in a member, can be expressed as a linear combination of the nodal displacements. i.e.

$$s(t) = \mathbf{l}^{\mathrm{T}} \mathbf{u}(t) = \mathbf{l}^{\mathrm{T}} \mathbf{\Phi} \mathbf{q}(t) \tag{20}$$

It is well recognized that the mean of the peak structural response $s(t)$ can be approximately expressed as a combination of the mean values of maximum modal responses. Each maximum modal response is obtained in terms of the ordinate of the mean response spectrum, associated with the corresponding modal frequency and damping factor. The combination coefficients of the modal responses for obtaining the nodal peak response are usually derived by assuming:[4,6] (i) the input process as a Gaussian white noise; (ii) the mean value of the peak of the structural response proportional to its standard deviation by means of a coefficient called peak factor; (iii) the peak factor to be approximately the same for the response of interest $s(t)$ and modal responses. It follows that, by means of the assumption of white noise input (in this case $\chi_j = \gamma_j$), after very simple algebra, the stationary variance of $s(t)$, by taking into account equations (18)–(20), can be written as follows:

$$E[s^2(t)] = \sum_{j=1}^{m} \sum_{k=1}^{m} \{a_j a_k \rho_{0,jk} + \gamma_k a_j c_k \rho_{1,jk} + \gamma_j a_k c_j \rho_{1,kj} + \gamma_j \gamma_k c_j c_k \rho_{2,jk}\} \sigma_{d_j} \sigma_{d_k} \tag{21}$$

where

$$a_j = (\mathbf{l}^{\mathrm{T}} \mathbf{\Phi} \quad \mathbf{0}^{\mathrm{T}}) \mathbf{L}_j \mathbf{p}, \quad c_j = (\mathbf{l}^{\mathrm{T}} \mathbf{\Phi} \quad \mathbf{0}^{\mathrm{T}}) \mathbf{S}_j \mathbf{p} \tag{22}$$

0 being the zero vector of order $(m \times 1)$.

Hence the mean value of the peak of the structural response can be written as follows:

$$\max|s(t)| = \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{m} e_j e_k \rho_{jk} D(\beta_j, \gamma_j) D(\beta_k, \gamma_k)} \tag{23}$$

where $D(\beta_j, \gamma_j)$ is the conventional response spectrum representing the mean peak response of the oscillator characterized by the frequency radian γ_j and the damping ratio β_j, defined into equation (14) as functions of the complex eigenvalues of the dynamical system, while ρ_{jk} is the correlation coefficient given by

$$\rho_{jk} = \{a_j a_k \rho_{0,jk} + \gamma_k a_j c_k \rho_{1,jk} + \gamma_j a_k c_j \rho_{1,kj} + \gamma_j \gamma_k c_j c_k \rho_{2,jk}\}/(e_j e_k) \tag{24}$$

The normalizing coefficients e_j appearing into equations (23) and (24) are given by the following equation:

$$e_j = \sqrt{a_j^2 + \gamma_j^2 c_j^2} \tag{25}$$

The e_j coefficients have been chosen in such a way that $\rho_{jk} = 1$ for $j = k$; in this case equation (23) leads to

$$\max|s(t)| = \sqrt{\sum_{i=1}^{m} e_i^2 D^2(\beta_i, \gamma_i)} \tag{26}$$

which coincides with the traditional SRSS (Square Root of the Squares Sum) combination rule of maxima.

It is important to note that, by adopting equation (23), it is possible to derive both modified CQC (Complete Quadratic Combination)[15] and modified Rosenblueth nodal[14] formulations. Indeed the first one, proposed by Sinha and Igusa[15], assumes that $\rho_{1,jk} = 0$ and $\rho_{2,jk} = \rho_{0,jk}$. While in the approach proposed by Villaverde[14] besides the assumption $\rho_{1,jk} = 0$ the white noise with limited length is considered.

3. PROPOSED APPROACH

3.1. *Evaluation of the covariances*

The previously described approach for the evaluation of the covariances of the response requires the solution of the eigenproblem (9) which leads to complex eigenproperties. Moreover, the real matrices and coefficients appearing in the combination rule (23) are evaluated by using the complex algebra. Here an alternative approach able to evaluate the covariances of the response without requiring the solution of the eigenproblem (9) is presented. Hence the covariances of the response are evaluated making use of real quantities only. The input is assumed to be a white noise process, as considered by the other approaches available in literature. However the formulation here presented for the evaluation of the cross-correlation coefficients could be extended to the case of filtered input.

In order to describe this alternative approach let us rewrite equation (7) as follows:[17,20]

$$\dot{\mathbf{z}}(t) = (\mathbf{D}_0 + \mathbf{D}_1)\mathbf{z}(t) + \mathbf{V}\mathbf{p}\ddot{u}_g(t) \tag{27}$$

in which

$$\mathbf{D}_0 = \begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ -\boldsymbol{\Omega}^2 & -\boldsymbol{\Xi}_\mathrm{d} \end{pmatrix}; \quad \mathbf{D}_1 = \begin{pmatrix} \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & -\boldsymbol{\Xi}_\mathrm{f} \end{pmatrix} \tag{28}$$

In these equations $\boldsymbol{\Xi}_\mathrm{d}$ is a diagonal matrix whose non-zero elements are the elements on the principal diagonal of $\boldsymbol{\Xi}$, while $\boldsymbol{\Xi}_\mathrm{f}$ is a symmetric matrix having zero elements along the principal diagonal and the off-diagonal elements equal to the corresponding off-elements of matrix $\boldsymbol{\Xi}$.

Hence it is: $\boldsymbol{\Xi}_{\mathrm{f}} = \boldsymbol{\Xi} - \boldsymbol{\Xi}_{\mathrm{d}}$. Obviously for classically damped systems $\mathbf{D}_1 = \mathbf{0}$ and $\boldsymbol{\Xi}_{\mathrm{f}} = \mathbf{0}$. Note that the matrix $\mathbf{D}_0$ defined in equation (28) can be regarded as the dynamical matrix of a classically damped system having the ith natural circular frequency $\omega_i = \Omega_{ii}$ and the ith damping ratio $\xi_i = \Xi_{ii}/(2\Omega_{ii})$.

By using the Kronecker algebra, it can be shown that, when $\ddot{u}_{\mathrm{g}}(t)$ is a Gaussian white noise input, the stationary response covariances can be evaluated as follows:[20]

$$\mathbf{E}[\mathbf{z}^{[2]}(t)] = -2\pi S_0 \mathbf{D}_2^{-1} \mathbf{V}^{[2]} \mathbf{p}^{[2]} \tag{29}$$

where S_0 is the power spectral density of the white noise and $\mathbf{D}_2$ is the following matrix:

$$\mathbf{D}_2 = \mathbf{D} \otimes \mathbf{I}_{2m} + \mathbf{I}_{2m} \otimes \mathbf{D} \tag{30}$$

By considering that $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1$, we can write $\mathbf{D}_2$ as follows:

$$\mathbf{D}_2 = \mathbf{D}_{2,0} + \mathbf{D}_{2,1} = \mathbf{D}_{2,0}(\mathbf{I}_{2m}^{[2]} + \mathbf{D}_{2,0}^{-1}\mathbf{D}_{2,1}) \tag{31}$$

where

$$\mathbf{D}_{2,0} = \mathbf{D}_0 \otimes \mathbf{I}_{2m} + \mathbf{I}_{2m} \otimes \mathbf{D}_0, \quad \mathbf{D}_{2,1} = \mathbf{D}_1 \otimes \mathbf{I}_{2m} + \mathbf{I}_{2m} \otimes \mathbf{D}_1 \tag{32}$$

Substitution of equation (31) into equation (29) leads to

$$E[\mathbf{z}^{[2]}(t)] = \mathbf{B} E[\mathbf{z}_0^{[2]}(t)] \tag{33}$$

where

$$\mathbf{B} = (\mathbf{I}_{2m}^{[2]} + \mathbf{D}_{2,0}^{-1}\mathbf{D}_{2,1})^{-1} \tag{34}$$

and $E[\mathbf{z}_0^{[2]}]$ is a vector listing the stationary covariances of the modal state variables of the classically damped system which are governed by the following differential equations:

$$\ddot{\mathbf{q}}_0(t) + \boldsymbol{\Xi}_{\mathrm{d}} \dot{\mathbf{q}}_0(t) + \boldsymbol{\Omega}^2 \mathbf{q}_0(t) = \mathbf{p}\ddot{u}_{\mathrm{g}}(t) \tag{35}$$

Hence $\mathbf{z}_0^{\mathrm{T}} = (\mathbf{q}_0^{\mathrm{T}} \quad \dot{\mathbf{q}}_0^{\mathrm{T}})$. The covariances vector $E[\mathbf{z}_0^{[2]}]$ can be obtained by means of the following simple relationship:

$$E[\mathbf{z}_0^{[2]}] = -2\pi S_0 \mathbf{D}_{2,0}^{-1} \mathbf{V}^{[2]} \mathbf{p}^{[2]} \tag{36}$$

If the Kronecker product $\mathbf{z}_0(t) \otimes \mathbf{z}_0(t)$ is inserted into equation (33) in such a way that the following relationship holds:

$$E[\mathbf{z}_0(t) \odot \mathbf{z}_0(t)] = E\left[\begin{pmatrix} \mathbf{q}_0^{[2]}(t) \\ \mathbf{q}_0(t) \otimes \dot{\mathbf{q}}_0(t) \\ \dot{\mathbf{q}}_0(t) \otimes \mathbf{q}_0(t) \\ \dot{\mathbf{q}}_0^{[2]}(t) \end{pmatrix}\right] \tag{37}$$

where the symbol $\odot$ means block Kronecker product (see Appendix), then equation (33) can be rewritten as follows:

$$\begin{aligned} E[\mathbf{z}^{[2]}(t)] = {} & \mathbf{B}_0 E[\mathbf{q}_0^{[2]}(t)] + \mathbf{B}_1 E[\mathbf{q}_0(t) \otimes \dot{\mathbf{q}}_0(t)] + \mathbf{B}_2 E[\dot{\mathbf{q}}_0(t) \otimes \mathbf{q}_0(t)] \\ & + \mathbf{B}_3 E[\dot{\mathbf{q}}_0(t) \otimes \dot{\mathbf{q}}_0(t)] \end{aligned} \tag{38}$$

in which the matrices $\mathbf{B}_j$ are defined by means of a partition of order $(4m^2 \times m^2)$ of the matrix $\mathbf{B}$, as follows:

$$\mathbf{B} = (\mathbf{B}_0,\ \mathbf{B}_1,\ \mathbf{B}_2,\ \mathbf{B}_3) \tag{39}$$

It is easy to show that the covariances introduced into equation (37) can be related to the covariances of the responses of the following differential equations:

$$\ddot{g}_k(t) + 2\xi_k\omega_k\dot{g}_k(t) + \omega_k^2 g_k(t) = \ddot{u}_{\mathrm{g}}(t), \quad g_k(0) = 0,\ \dot{g}_k(0) = 0; \quad k = 1, 2, \ldots, m \tag{40}$$

by means of the following expressions:

$$\begin{aligned} E[q_{0,j}(t)q_{0,k}(t)] &= p_j p_k E[g_j(t)g_k(t)], \quad E[q_{0,j}(t)\dot{q}_{0,k}(t)] = p_j p_k E[g_j(t)\dot{g}_k(t)];\\ E[\dot{q}_{0,j}(t)q_{0,k}(t)] &= p_j p_k E[\dot{g}_j(t)g_k(t)], \quad E[\dot{q}_{0,j}(t)\dot{q}_{0,k}(t)] = p_j p_k E[\dot{g}_j(t)\dot{g}_k(t)] \end{aligned} \tag{41}$$

In this way, equation (38) gives

$$\begin{aligned} E[\mathbf{z}^{[2]}(t)] = \sum_{j=1}^{m}\sum_{k=1}^{m} &\mathbf{b}_{0,jk}E[g_j(t)g_k(t)]\,p_jp_k + \mathbf{b}_{1,jk}E[g_j(t)\dot{g}_k(t)]\,p_jp_k\\ &+ \mathbf{b}_{2,jk}E[\dot{g}_j(t)g_k(t)]\,p_jp_k + \mathbf{b}_{3,jk}E[\dot{g}_j(t)\dot{g}_k(t)]\,p_jp_k \end{aligned} \tag{42}$$

where $\mathbf{b}_{i,jk}$ (with $i = 0, 1, 2, 3$) is the $[\,j + (k-1)m]$th column of the matrix $\mathbf{B}_i$. It is important to note that equation (42) gives the modal covariances, without any approximation, exactly as equation (15).

By comparing equations (13) and (40), we observe that in the first one the radian frequency γ_k and the damping ratio β_k are evaluated as functions of complex eigenvalues, while in the second one the corresponding quantities ω_k and ξ_k are evaluated by solving only the real eigenproblem (3).

Moreover, by introducing the following cross-correlation coefficients

$$\bar{\rho}_{0,jk} = \frac{E[g_jg_k]}{\sigma_{g_j}\sigma_{g_k}}, \quad \bar{\rho}_{1,jk} = \frac{E[g_j\dot{g}_k]}{\sigma_{g_j}\sigma_{\dot{g}_k}}, \quad \bar{\rho}_{2,jk} = \frac{E[\dot{g}_j\dot{g}_k]}{\sigma_{\dot{g}_j}\sigma_{\dot{g}_k}} \tag{43}$$

and, taking into account that for white noise input is $\sigma_{\dot{g}_j} = \omega_j\sigma_{g_j}$, lastly we can write

$$E[\mathbf{z}^{[2]}(t)] = \sum_{j=1}^{m}\sum_{k=1}^{m} \bar{\mathbf{r}}_{jk}p_jp_k\sigma_{g_j}\sigma_{g_k} \tag{44}$$

where $\bar{\mathbf{r}}_{jk}$ is a vector of order $4m^2$ given as follows:

$$\bar{\mathbf{r}}_{jk} = \mathbf{b}_{0,jk}\bar{\rho}_{0,jk} + \omega_k\mathbf{b}_{1,jk}\bar{\rho}_{1,jk} + \omega_j\mathbf{b}_{2,jk}\bar{\rho}_{1,kj} + \omega_j\omega_k\mathbf{b}_{3,jk}\bar{\rho}_{2,jk} \tag{45}$$

It is important to note that, by applying the proposed procedure, the modal response can be evaluated without solving complex eigenproblems, but only solving the traditional real eigenproblem (3) of a classically damped system. It follows that by applying this procedure it is possible to reduce drastically the computational effort.

Once the modal covariances $E[\mathbf{z}^{[2]}(t)]$ are evaluated, the nodal ones can be obtained by means of the following relationship:

$$E\left[\begin{pmatrix}\mathbf{u}(t)\\ \dot{\mathbf{u}}(t)\end{pmatrix}^{[2]}\right] = \begin{pmatrix}\mathbf{\Phi} & \mathbf{0}_m\\ \mathbf{0}_m & \mathbf{\Phi}\end{pmatrix}^{[2]} E[\mathbf{z}^{[2]}(t)] \tag{46}$$

where the block Kronecker product has to be applied.

3.2. Evaluation of the peak response

It can be easily shown that, if the variances of the modal response are evaluated by means of equation (44), then the stationary variance of the reference structural response $s(t)$ assumes the following form:

$$E[s^2(t)] = \sum_{j=1}^{m}\sum_{k=1}^{m}[b_{0,jk}\bar{\rho}_{0,jk} + \omega_k b_{1,jk}\bar{\rho}_{1,jk} + \omega_j b_{2,jk}\bar{\rho}_{1,kj} + \omega_j\omega_k b_{3,jk}\bar{\rho}_{2,jk}]\sigma_{g_j}\sigma_{g_k}p_j p_k \quad (47)$$

where

$$b_{i,jk} = (\mathbf{l}^{\mathrm{T}}\mathbf{\Phi} \quad \mathbf{0}^{\mathrm{T}})^{[2]}\mathbf{b}_{i,jk}; \quad i = 0, 1, 2, 3 \quad (48)$$

and the block Kronecker product has to be applied. Hence, by assuming the peak factor related to $s(t)$ as those related to the modal responses $g_j(t)$, the maximum of $s(t)$ can be estimated as follows:

$$\max|s(t)| = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m}\bar{\rho}_{jk}p_j p_k \bar{D}(\xi_j, \omega_j)\bar{D}(\xi_k, \omega_k)} \quad (49)$$

where $\bar{D}(\xi_j, \omega_j)$ is given by the ordinate of the mean peak displacement spectrum corresponding to the radian frequency ω_j and the damping ratio ξ_j. The cross-correlation coefficients appearing in equation (49) are given by

$$\bar{\rho}_{jk} = b_{0,jk}\bar{\rho}_{0,jk} + \omega_k b_{1,jk}\bar{\rho}_{1,jk} + \omega_j b_{2,jk}\bar{\rho}_{1,kj} + \omega_j\omega_k b_{3,jk}\bar{\rho}_{2,jk} \quad (50)$$

In this way a modal combination rule for non-classically damped system, which does not require the solution of any complex eigenproblem, has been introduced. Similar to methods using the complex analysis, the presented approach suffers the draw-back connected to the requirement of various response spectra for different values of damping ratio. This fact can represent a problem because the response spectra are usually given for some particular values of damping ratios and only approximated relationships are available in order to obtain the response spectra for different values of the damping ratio. In the following section of modification of the presented approach is presented in order to avoid this inconvenience.

4. PROPOSED APPROACH WITH FIXED DAMPING RATIO

In all the national and international codes the reference response spectra are given for fixed values of the damping ratio (the so-called target response spectra are usually given for $\xi = 0{\cdot}05$ and in some cases for $\xi = 0{\cdot}01$, 0·02 and 0·1) and suitable relationships are provided in order to obtain the corresponding spectrum for different values of damping ratios. Unfortunately the spectra obtained by means of these relationships show, in some cases, a very poor accuracy. It is evident that all the modal combination rules available for non-classically damped systems, included the approach presented in the previous sections are affected by this problem. In order to overcome this drawback in this section a modification of the approach presented in the previous section is proposed.

To this purpose let us rewrite equation (27) as follows:

$$\dot{\mathbf{z}}(t) = (\hat{\mathbf{D}}_0 + \hat{\mathbf{D}}_1)\mathbf{z}(t) + \mathbf{V}\mathbf{p}\ddot{u}_g(t) \tag{51}$$

where now

$$\hat{\mathbf{D}}_0 = \begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ -\boldsymbol{\Omega}^2 & -2\xi\boldsymbol{\Omega} \end{pmatrix}, \quad \hat{\mathbf{D}}_1 = \begin{pmatrix} \mathbf{0}_m & \mathbf{0}_m \\ \mathbf{0}_m & -\hat{\boldsymbol{\Xi}}_f \end{pmatrix} \tag{52}$$

in which ξ is the value of the damping ratio chosen by the reference displacement spectrum and $\hat{\boldsymbol{\Xi}}_f = \boldsymbol{\Xi} - 2\xi\boldsymbol{\Omega}$.

Application of the procedure shown in the previous section to equation (51) leads to the following expression for the modal combination:

$$\max|s(t)| = \sqrt{\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{\rho}_{jk} p_j p_k D(\xi, \omega_j) D(\xi, \omega_k)} \tag{53}$$

where

$$\hat{\rho}_{jk} = \hat{b}_{0,jk}\hat{\rho}_{0,jk} + \omega_k \hat{b}_{1,jk}\hat{\rho}_{1,jk} + \omega_j \hat{b}_{2,jk}\hat{\rho}_{1,kj} + \omega_j\omega_k \hat{b}_{3,jk}\hat{\rho}_{2,jk} \tag{54}$$

where the cross-correlation coefficients $\hat{\rho}_{i,jk}$ (with $i = 0, 1, 2$) are those related to SDOF oscillators characterized by the value ξ of the damping ratio and by the radian frequencies ω_j and ω_k. In equation (54) the terms $\hat{b}_{i,jk}$ (with $i = 0, 1, 2, 3$) are related, by means of relationships analogous to those given in equations (39) and (48), to the matrix $\hat{\mathbf{B}}$ given by

$$\hat{\mathbf{B}} = (\mathbf{I}_{2m}^{[2]} + \hat{\mathbf{D}}_{2,0}^{-1}\hat{\mathbf{D}}_{2,1})^{-1} \tag{55}$$

in which

$$\hat{\mathbf{D}}_{2,0} = \hat{\mathbf{D}}_0 \odot \mathbf{I}_{2m} + \mathbf{I}_{2m} \odot \hat{\mathbf{D}}_0, \quad \hat{\mathbf{D}}_{2,1} = \hat{\mathbf{D}}_1 \odot \mathbf{I}_{2m} + \mathbf{I}_{2m} \odot \hat{\mathbf{D}}_1 \tag{56}$$

The modal combination rule expressed by means of equation (53) allows us to consider the non-classically damped systems as the classically damped ones characterized by a given value of the damping ratio. This can be very useful in the design codes of this kind of structures.

5. NUMERICAL EXAMPLES

Since the proposed procedures give the exact response variances of MDOF structural systems subjected to a base excitation approximated as a white noise process, in this section the attention is devoted to the study of the accuracy in the evaluation of the cross-correlation coefficients and in the use of the modal combination rules. In particular the maximum peak of a response quantity of interest is obtained by using the proposed methods, and the results are compared with those obtained by other methods proposed in literature.

The approximated results are compared with a corresponding reference value that is an estimation of the maximum value of the response quantity of interest. Here this is obtained by means of the following steps: (a) evaluation of the exact response power spectral density of the response quantity of interest; it is obtained exactly, without the use of any modal transformation, under the assumption of white noise input; (b) evaluation of the spectral moments and of the peak factor by applying the well-known relationship introduced by Vanmarcke;[22] in particular a fractile $p = 0{\cdot}99$ and a duration time $T = 20$ s have been considered; (c) estimation of the maximum value by means of the relationship given in the above cited Vanmarcke's work.[22] In this way the evaluation of the reference maximum value is not affected by the approximation characterizing all the modal combination rules, including those presented here, since the peak coefficients are equal.

The displacement maximum value of a SDOF oscillator excited by a white noise and characterized by a radian frequency ω and a damping ratio ξ, evaluated by the previous steps (a)–(c), is here considered as response spectrum value $D(\xi, \omega)$. Alternatively, these quantities could be obtained by Monte Carlo simulations. It is clear that, when the procedures considered are applied in practice, the value of $D(\xi, \omega)$ is that fixed by the response spectrum used for representing the earthquake. Hence the steps (a)–(c) before cited are necessary only for evaluating the reference value and the response spectrum values $D(\xi, \omega)$, in order to test the approximate procedures here proposed, but they are not necessary for using the modal combination rules.

It is important to note that the modal maximum responses obviously depend on the values of the modal radian frequency and damping ratio; and that these values are different depending on the kind of modal combination used. In particular, if the Shina–Igusa combination rule is used, these values are respectively γ_j and β_j (given into equations (14)). On the other hand, if the modal combination rule introduced in Section 3 is used, these values are ω_j and ξ_j; at last, in the alternative approach presented in Section 4, the value of the radian frequencies are ω_j, while the damping ratio is the fixed value considered in the response spectrum; here in particular it was chosen $\xi = 0{\cdot}05$.

5.1. Example 1

The simplest system having non-classical damping is the 2-DOF system represented in Figure 1. This system was extensively studied in the past. In particular, Crandall and Mark[21] used the frequency domain approach to find the stochastic characteristics of the response of this system subjected to a white noise excitation, while Igusa *et al.*[7] used the modal decomposition method for the same objective. The study of this system is very useful because it is the model of two representative cases of non-classically damped structural 2-DOF systems: the soil–structure system and the equipment–structure system. In the first case the damping ratios of the soil (substructure 1) and of the structure (substructure 2) are quite different; in the second case the mass ratio m_2/m_1 is very small, the frequency ratio is close to 1 and the difference of the damping ratios is such that

$$(\xi_1 - \xi_2)^2 > m_2/m_1 \tag{57}$$

Furthermore, the choice of this example allows one to find the dependence of the results on the values of the coupling factor α (defined in equation (6)) or of similar quantities representative of the level of the system non-classicity. Finally in this example, for some suitable values of the system parameters, since the modal radian frequencies are very close, the SRSS modal combination rule does not give accurate results.

In order to illustrate the good accuracy of the proposed modal combination rule with respect to the other ones presented in literature, for the above-considered system a parametric study has been conducted, in the two cases of soil–structure and equipment–structure systems, for different values of damping ratio differences. The chosen values of system parameters are just those considered in the work of Igusa *et al.*[7], that is: for the case of soil–structure system: average damping ratio $\xi_a = (\xi_1 - \xi_2)/2 = 0{\cdot}2$, mass ratio $m_2/m_1 = 0{\cdot}3$ and natural circular frequencies

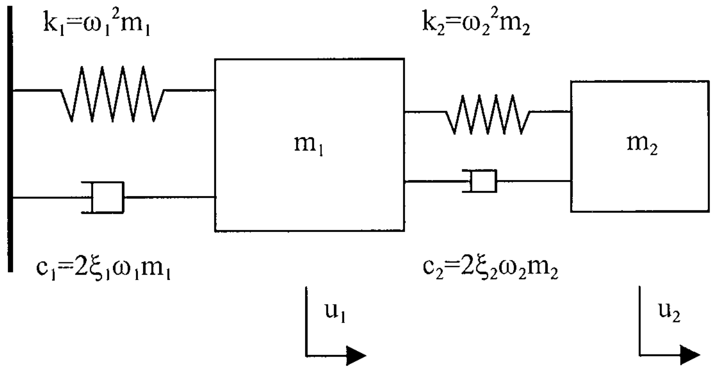

Figure 1. Example 1: two-DOF system considered

$\omega_1 = \omega_2 = 10{\cdot}0$ rad/s; while for the case of the equipment–structure system: $\xi_a = 0{\cdot}04$, $m_2/m_1 = 0{\cdot}001$ and $\omega_1 = \omega_2 = 10$ rad/s. In both systems the varying parameter is the difference of damping ratios $\xi_1 - \xi_2$. The excitation considered is a unitary white noise. The reference nodal response considered is the relative displacement between the two masses $s(t) = u_2(t) - u_1(t)$.

In Figures 2 and 3 the results in terms of the maximum value of $s(t)$ versus the damping ratio $\xi_1 - \xi_2$ (which is strictly related to the value of the coupling index α) have been reported. From the analysis of these figures the great accuracy of the proposed approach, related to the fact that no complex eigenvalue problem has to be solved, the advantages in applying this procedure are evident. Moreover the results of the presented procedure do not change substantially in the case of fixed modal damping ratios and this has been proved for various values of damping ratio. This is a very important result for the applicability of this modal combination rule in the design codes of the non-classically damped structures.

5.2. Example 2

In this example another important class of non-classically damped systems represented by the isolated buildings is treated. In particular the same five-storey shear-frame considered in Example 3 of the 1996 Chopra work[23] and in Chapter 20 of his book[1] is taken into account. As the modal natural circular frequencies are very different from each other, the influence of the modal cross combinations is very limited and the SRSS and the CQC modal combination rules lead practically to the same results. The fundamental goal of this application is to investigate the accuracy of the proposed procedures for this kind of structures, too. The results are reported in Table I and are referred to two nodal reference peak responses: the displacement $u_5(t)$ of the last storey (referred to the isolation system), and the relative displacement between the last and the fourth storey $u_5(t) - u_4(t)$. The input is considered to be a unit white noise process, again. From these results the excellent accuracy of the procedures is evidenced. In particular, it is important to evidence that the procedures proposed in Sections 3 and 4 lead to the same results, which confirms that, in the presented combination rule the value of the reference maximum nodal response does not depend on the value of the modal spectrum dampings considered. A change in

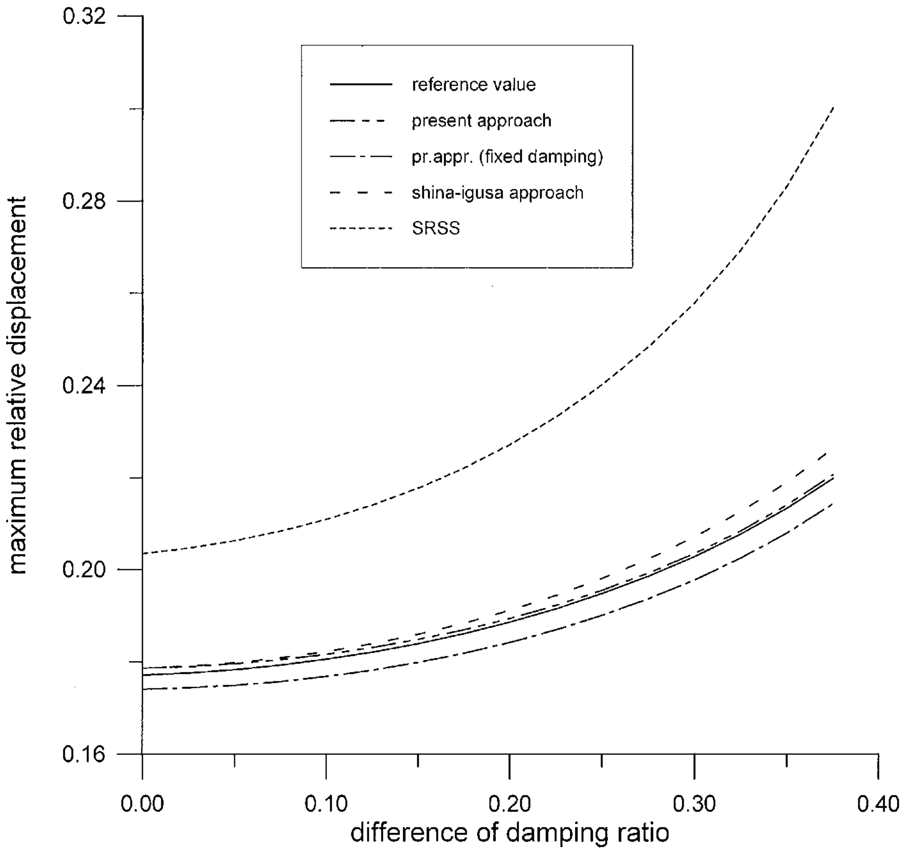

Figure 2. Maximum nodal relative displacement in the first case of Example 1 ($\xi_a = 0{\cdot}2$, $m_2/m_1 = 0{\cdot}3$, $\omega_1 = \omega_2 = 10$ rad/s)

the modal spectrum dampings implies a corresponding change in the cross-correlation coefficients in such a way that the estimation of the maximum value of the interested response quantity remains unchanged.

Hence, in this way, the response spectrum approach can be applied for isolated building too, by using only target response spectra.

6. CONCLUSIONS

In this work a method for the evaluation of the cross-correlation coefficients of MDOF non-classically damped structural systems is proposed. The method is developed in the time domain and requires neither the solution of the complex eigenproblem related to the dynamical matrix of the structure nor the velocity or cosine spectra. Indeed an opportune classically damped system is considered for the evaluation of the cross-correlation coefficients; in this way all the necessary quantities are real. Besides this computational advantage, the proposed approach

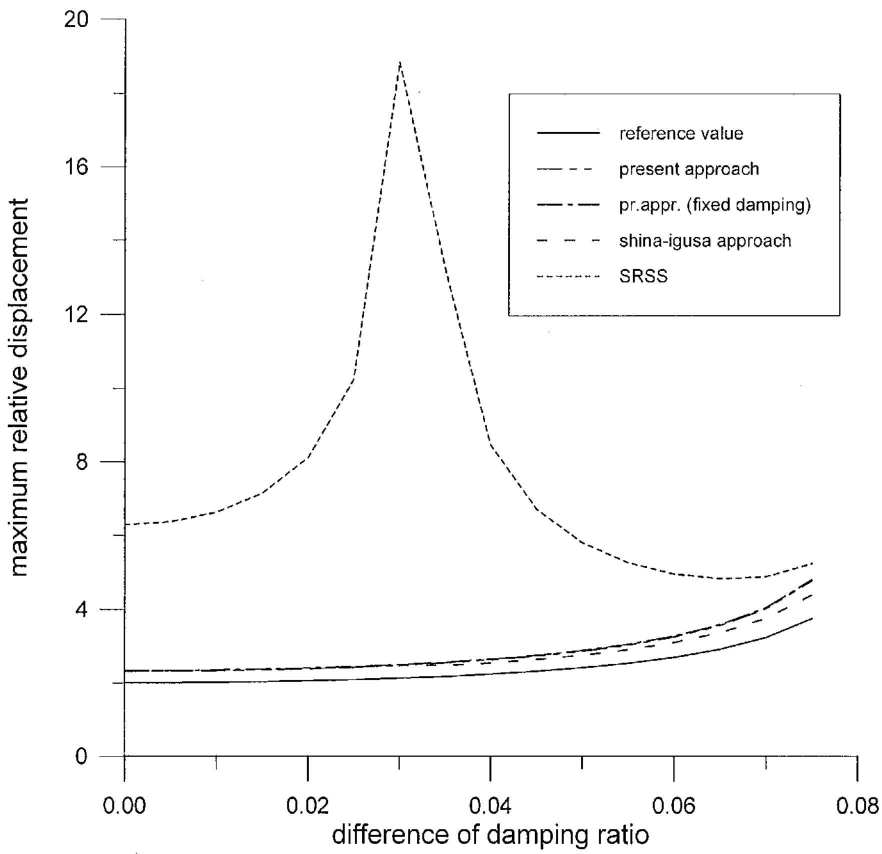

Figure 3. Maximum nodal relative displacement in the second case of Example 1. ($\xi_a = 0{\cdot}04$, $m_2/m_1 = 0{\cdot}001$, $\omega_1 = \omega_2 = 50$ rad/s)

Table I. Example 2: maximum value of the nodal response quantity of interest: comparison among the reference results and the approximated ones obtained by the proposed CQC approaches, the Shina–Igusa approach and the SRSS approach

	Reference	pr.appr. (3)	pr.appr. (4)	S.I. appr.	SRSS
u_5	0·1865	0·1861	0·1862	0·1824	0·1824
u_5–u_4	0·0126	0·0126	0·0125	0·0123	0·0123

exhibits another important feature. In fact it is shown that, when the modal combination rule is used for the estimation of the structural response maxima, the results do not change if, in the above cited classical damped system, the value of the damping ratios is fixed. Hence the proposed approach presents the advantage that any target response spectrum can be considered in the design of the non-classically damped structures, without the use of any approximated relationship transforming the spectrum in order to take into account the different values of the damping ratios. This fact implies a great accuracy in the results obtained by using the proposed procedure, which is evident by the results reported in the application.

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APPENDIX

In this appendix the operation between two matrix quantities called *block Kronecker product* and introduced in Section 3 is explained.

It is well known that, given two matrices $\mathbf{A}$ and $\mathbf{B}$, or order $(m \times n)$ and $(p \times q)$, respectively, the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is the matrix $\mathbf{C}$, or order $(mp \times nq)$, obtained by multiplying each element of $\mathbf{A}$ for all the matrix $\mathbf{B}$ in such a way that:[24]

$$\mathbf{C} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \tag{58}$$

Now, if $\mathbf{A}$ and $\mathbf{B}$ are supermatrices built by submatrices blocks by

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \tag{59}$$

then, beyond the classical Kronecker product as introduced into equation (58), it is sometimes useful to introduce the block Kronecker product which is defined as follows:

$$\mathbf{A} \odot \mathbf{B} = \begin{pmatrix} \mathbf{A}_{11} \otimes \mathbf{B}_{11} & \mathbf{A}_{11} \otimes \mathbf{B}_{12} & \mathbf{A}_{12} \otimes \mathbf{B}_{11} & \mathbf{A}_{12} \otimes \mathbf{B}_{12} \\ \mathbf{A}_{11} \otimes \mathbf{B}_{21} & \mathbf{A}_{11} \otimes \mathbf{B}_{22} & \mathbf{A}_{12} \otimes \mathbf{B}_{21} & \mathbf{A}_{12} \otimes \mathbf{B}_{22} \\ \mathbf{A}_{21} \otimes \mathbf{B}_{11} & \mathbf{A}_{21} \otimes \mathbf{B}_{12} & \mathbf{A}_{22} \otimes \mathbf{B}_{11} & \mathbf{A}_{22} \otimes \mathbf{B}_{12} \\ \mathbf{A}_{21} \otimes \mathbf{B}_{21} & \mathbf{A}_{21} \otimes \mathbf{B}_{22} & \mathbf{A}_{22} \otimes \mathbf{B}_{21} & \mathbf{A}_{22} \otimes \mathbf{B}_{22} \end{pmatrix} \tag{60}$$

It is obvious that this definition, here introduced for two (2×2) block matrices, can be extended to any order block matrices.

REFERENCES

1. A. K. Chopra, *Dynamics of Structures—Theory and Applications to Earthquake Engineering*, Prentice-Hall, Englewood Cliffs, NJ 1995.
2. W. C. Hurty and M. F. Rubinstein, *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, NJ, 1964.
3. R. W. Traill-Nash, 'Modal methods in the dynamics of systems with non-classical damping', *Earthquake Engng. Struct. Dyn.* **9**, 153–169 (1981).
4. A. K. Gupta, *Response Spectrum Method in Seismic Analysis of Design of Structures*, Blackwell Scientific Publications, Boston, 1990.

5. A. Der Kiureghian, 'Structural response to stationary excitation', *J. Engng. Mech. Div. ASCE* **106**(6), 1195–1213 (1980).
6. E. L. Wilson, A. Der Kiureghian and E. P. Bayo, 'A replacement for the SRSS method for seismic analysis', *Earthquake Engng. Struct. Dyn.* **9**, 187–194 (1981).
7. T. Igusa, A. Der Kiureghian and J. L. Sackman, 'Modal decomposition method for stationary response of non-classically damped systems', *Earthquake Engng. Struct. Dyn.* **12**, 121–136 (1984).
8. A. K. Gupta and J. W. Jaw, 'Seismic response of non classically damped structures', *Nuclear Engng. Design* **91**, 153–159 (1986).
9. G. Muscolino, 'Dynamic response of multiply connected primary-secondary systems', *Earthquake Engng. Struct. Dyn.* **19**, 205–216 (1990).
10. A. Der Kiureghian, 'A response spectrum method approach for random vibration of MDOF systems', *Earthquake Engng. Struct. Dyn.* **9**, 419–435 (1981).
11. M. P. Singh, 'Seismic response by SRSS for nonproportional damping', *J. Engng. Mech. Div. ASCE* **106**(6), 1405–1419 (1980).
12. A. K. Gupta and J. W. Jaw, 'Response spectrum method for nonclassically damped systems', *Nucl. Engng. Des.* **91**, 161–169 (1986).
13. J. N. Yang, S. Sarkani and F. X. Long, 'A response spectrum approach for seismic analysis of nonclassically damped structures', *Engng. Struct.* **12**, 173–184 (1990).
14. R. Villaverde, 'Rosemblueth's modal combination rule for systems with non-classical damping', *Earthquake Engng. Structu. Dyn.* **16**, 315–328 (1988).
15. R. Sinha and T. Igusa, 'CQC and SRSS method for non-classically damped structures', *Earthquake Engng. Struct. Dyn.* **24**, 615–619 (1995).
16. S. T. Mau, 'A subspace modal superposition method for non-classically damped systems', *Earthquake Engng. Struct. Dyn.* **16**, 931–942 (1988).
17. A. D'Aveni and G. Muscolino, 'Response of non-classically damped structures in the modal subspace', *Earthquake Engng. Struct. Dyn.* **24**, 1267–1281 (1995).
18. A. M. Claret and F. Venancio-Filho, 'A modal superposition pseudo-force method for dynamic analysis of structural systems with non-proportional damping', *Earthquake Engng. Struct. Dyn* **20**, 303–315 (1991).
19. G. Muscolino, 'Non-stationary pre-envelope covariance of non-classically damped systems', *J. Sound Vibr.* **149**(1), 107–123 (1991).
20. G. Muscolino, 'Dynamically modified linear structures: deterministic and stochastic response', *J. Engng. Mech. ASCE* **122**(11), 1044–1051 (1996).
21. S. H. Crandall and W. D. Mark, *Random Vibration of Mechanical Systems*, Academic Press, New York, 1963.
22. E. H. Vanmarcke, 'Structural response to earthquakes', in C. Lomnitz and E. Rosenblueth (eds), *Seismic Risk and Engineering Decisions*, Chap. 8, Elsevier, Amsterdam, 1978.
23. A. K. Chopra, 'Modal analysis of linear dynamic systems: physical interpretation', *J. Struct. Engng.* **122**, 517–527 (1996).
24. R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1974.

